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A Mehler–Heine-type formula for Hermite–Sobolev orthogonal polynomials ☆

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Abstract

We consider a Sobolev inner product such as

$$(f,g)_S = \int f(x)g(x)\,\mathrm{d}\mu_0(x) + \lambda \int f'(x)g'(x)\,\mathrm{d}\mu_1(x), \quad \lambda > 0, \tag{1}$$

with (μ_0, μ_1) being a symmetrically coherent pair of measures with unbounded support. Denote by Q_n the orthogonal polynomials with respect to (1) and they are so-called Hermite–Sobolev orthogonal polynomials. We give a Mehler–Heine-type formula for Q_n when μ_1 is the measure corresponding to Hermite weight on $\mathbb{R}$, that is, $\mathrm{d}\mu_1 = \mathrm{e}^{-x^2}\,\mathrm{d}x$ and as a consequence an asymptotic property of both the zeros and critical points of Q_n is obtained, illustrated by numerical examples. Some remarks and numerical experiments are carried out for $\mathrm{d}\mu_0 = \mathrm{e}^{-x^2}\,\mathrm{d}x$. An upper bound for $|Q_n|$ on $\mathbb{R}$ is also provided in both cases.

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Keywords: Sobolev orthogonal polynomials; Asymptotics; Mehler–Heine-type formulas

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1. Introduction

We consider the Sobolev inner product

$$(f,g)_S = \int f(x)g(x)\,\mathrm{d}\mu_0(x) + \lambda \int f'(x)g'(x)\,\mathrm{d}\mu_1(x), \quad \lambda > 0, \tag{2}$$

where μ_i, $i=1,2$ are positive Borel measures with support $I_i \subseteq \mathbb{R}$, respectively. Denote for $Q_n(x) = 2^n x^n + \cdots$, those polynomials that are orthogonal with respect to (2). The Sobolev orthogonal polynomials were introduced in [4] in connection with the least-squares simultaneous approximation of a function and its derivatives. In the early 1990s, Iserles et al. introduced in [3] the fruitful concept of coherent pair of measures and, for symmetric measures, symmetrically coherent pair. Later, Meijer gave in [6] a complete classification of all coherent pairs and symmetrically coherent pairs. In particular, it was established that at least one of the measures in each coherent pair has to be classic (i.e., Jacobi, Laguerre o Hermite). Thus, if one of the measures corresponds to Hermite weight function, $\mathrm{e}^{-x^2}\,\mathrm{d}x$ on $\mathbb{R}$, there are only two possibilities (see [6]):

(a) *Case* I. $\mathrm{d}\mu_0 = (x^2+a^2)\mathrm{e}^{-x^2}\,\mathrm{d}x$, $\mathrm{d}\mu_1 = \mathrm{e}^{-x^2}\,\mathrm{d}x$, $a \in \mathbb{R}$.

(b) *Case* II. $\mathrm{d}\mu_0 = \mathrm{e}^{-x^2}\,\mathrm{d}x$, $\mathrm{d}\mu_1 = \dfrac{\mathrm{e}^{-x^2}}{x^2+a^2}\,\mathrm{d}x$, $a \in \mathbb{R}\backslash\{0\}$.

Let (μ_0, μ_1) be a pair of measures of *Cases* I or II, then Q_n are so-called Hermite–Sobolev orthogonal polynomials. Analytic properties of these polynomials, such as asymptotics for $Q_n(x)$ in $\mathbb{C}\backslash\mathbb{R}$ or Plancherel–Rotach-type asymptotics in $\mathbb{C}\backslash[-\sqrt{2},\sqrt{2}]$, have been obtained in [2]. Also, in the aforementioned paper, the accumulation sets of the zeros of Q_n before and after an appropriate scaling of the plane are obtained.

On the other hand, we think that the Mehler–Heine-type formulas for Sobolev orthogonal polynomials are interesting, both analytically and numerically, since they are the natural way to establish a limit relation between these orthogonal polynomials and the well-known Bessel function $J_k(x)$ defined as (see e.g. [7, p. 15]):

$$J_k(x) = \sum_{j=0}^{\infty} \frac{(-1)^j (x/2)^{2j+k}}{j!\,\Gamma(j+k+1)}.$$

In this sense, a Mehler–Heine-type formula for the so-called non-diagonal Laguerre–Sobolev orthogonal polynomials has been obtained in [5].

Here, we look for a Mehler–Heine-type formula for the orthogonal polynomials Q_n. Thus, in the *Case* I, we give a limit relation between the appropriately scaled Hermite–Sobolev polynomials Q_n and the elementary trigonometric functions $\sin(x)$ and $\cos(x)$ (these function can be expressed, see [7, f.(1.71.2), p. 15], in terms of $J_{1/2}(x)$ and $J_{-1/2}(x)$, respectively). This result allows us to know the asymptotic behavior of Q_n (and of its derivatives) in the neighborhood of 0. As a consequence of this, we obtain an asymptotic property of the zeros and critical points of Q_n, supported by some numerical examples in Section 4. Also, we discuss some problems of *Case* II and, in Section 4 a conjecture about the small zeros of Q_n is done in this case, supported by numerical examples. Finally, in Section 3, we give an upper bound for $|Q_n|$ on $\mathbb{R}$, analogous to that for Hermite polynomials.

The notation that we use in this work: H_n denotes the Hermite polynomial orthogonal with respect to the inner product

$$(f,g) := \int_{-\infty}^{\infty} f(x)g(x)\mathrm{e}^{-x^2}\,\mathrm{d}x,$$

with the normalization $H_n(x) = 2^n x^n + \cdots$ and Q_n are chosen with the same leading coefficient. Finally, we also denote

$$\varphi(x) = x + \sqrt{x^2-1}, \quad k_n = (H_n, H_n), \ \tilde{k}_n = (Q_n, Q_n)_S.$$

On the other hand, in order to obtain Theorem 1, we use the well-known Mehler–Heine-type formula for Hermite polynomials, that is, for j fixed (see, for example, [1, p. 346] or, [7, p. 193] using the relation between Hermite and Laguerre polynomials) we have:

$$\lim_{n\to\infty} \frac{(-1)^n\sqrt{n+j}H_{2n}(x/(2\sqrt{n+j}))}{2^{2n}n!} = \frac{1}{\sqrt{\pi}}\cos(x), \tag{3}$$

$$\lim_{n\to\infty} \frac{(-1)^n H_{2n+1}(x/(2\sqrt{n+j}))}{2^{2n+1}n!} = \frac{1}{\sqrt{\pi}}\sin(x), \tag{4}$$

both uniformly on compact subsets of $\mathbb{C}$.

2. Mehler–Heine-type formula

First, we consider the *Case* I and so we have the inner product

$$(f,g)_S = \int_{-\infty}^{\infty} f(x)g(x)(x^2+a^2)\,\mathrm{e}^{-x^2}\,\mathrm{d}x + \lambda\int_{-\infty}^{\infty} f'(x)g'(x)\,\mathrm{e}^{-x^2}\,\mathrm{d}x. \quad \lambda > 0, \ a\in\mathbb{R}. \tag{5}$$

In this situation we get:

Theorem 1. *Let* $\Theta(\lambda) = \varphi(1+2\lambda)/(\varphi(1+2\lambda)-1)$. *The following Mehler–Heine-type formulas for the polynomials* $Q_n(x) = 2^n x^n + \cdots$ *orthogonal with respect to* (5) *hold*:

$$\lim_{n\to\infty} \frac{(-1)^n\sqrt{n}Q_{2n}(x/(2\sqrt{n}))}{2^{2n}n!} = \Theta(\lambda)\frac{\cos(x)}{\sqrt{\pi}},$$

$$\lim_{n\to\infty} \frac{(-1)^n Q_{2n+1}(x/(2\sqrt{n}))}{2^{2n+1}n!} = \Theta(\lambda)\frac{\sin(x)}{\sqrt{\pi}},$$

both uniformly on compact subsets of $\mathbb{C}$.

Proof. The polynomials H_n and Q_n satisfy the relation (see, for example, [2, Lemma 2.1]):

$$H_n = Q_n + a_{n-2}Q_{n-2}, \quad n \geqslant 0, \tag{6}$$

where $a_{n-2} = k_n/(4\tilde{k}_{n-2})$, $n \geqslant 2$, and $a_{-1} = a_{-2} = 0$. Applying (6) in a recursive way, we obtain

$$Q_m(x) = \sum_{i=0}^{[m/2]} (-1)^i b_i^{(m)} H_{m-2i}(x), \quad m \geqslant 0, \tag{7}$$

where

$$b_i^{(m)} = \prod_{j=1}^{i} a_{m-2j} \quad \text{for } i \geqslant 1 \text{ and } b_0^{(m)} = 1$$

and $[m]$ means the greatest integer less than or equal to m. First we consider m as even, that is, $m = 2n$. Then, scaling the variable x in (7) we can write

$$\begin{aligned}\frac{(-1)^n \sqrt{n} Q_{2n}(x/(2\sqrt{n}))}{2^{2n} n!} &= \sum_{i=0}^{n} \frac{b_i^{(2n)} \sqrt{n}}{2^{2i} \prod_{j=0}^{i-1} (n-j)} \frac{(-1)^{n-i} H_{2n-2i}(x/(2\sqrt{n}))}{2^{2n-2i}(n-i)!} \\ &:= \sum_{i=0}^{n} g_{n,i}(x/(2\sqrt{n})),\end{aligned} \tag{8}$$

where

$$g_{n,i}(x/(2\sqrt{n})) = (-1)^{n-i} c_i^{(2n)} \frac{\sqrt{n} H_{2n-2i}(x/(2\sqrt{n}))}{2^{2n-2i}(n-i)!},$$

with

$$c_i^{(2n)} = \frac{b_i^{(2n)}}{2^{2i} \prod_{j=0}^{i-1} (n-j)},$$

and the assumption $\prod_{j=0}^{-1} (n-j) = 1$. On the other hand, in [2, Lemma 2.2] it was established that the sequence $\{a_n/(2(n+2))\}$ is uniformly bounded by $r := 1/(1+2\lambda) < 1$ and

$$\lim_{n\to\infty} \frac{a_n}{2(n+2)} = \frac{1}{\varphi(1+2\lambda)}. \tag{9}$$

Thus, using the bound for $\{a_n/(2(n+2))\}$, we obtain, for $i = 0, \ldots n$, that $|c_i^{(2n)}| \leqslant r^i$. Now, if x belongs to a compact subset of $\mathbb{C}$, using (3), we have for n large enough and $0 \leqslant i \leqslant n$,

$$\left| \frac{\sqrt{n} H_{2n-2i}(x/(2\sqrt{n}))}{2^{2n-2i}(n-i)!} \right| \leqslant \mathscr{M},$$

where $\mathscr{M}$ is a constant and, therefore,

$$|g_{n,i}(x/(2\sqrt{n}))| \leqslant \mathscr{M} r^i. \tag{10}$$

Then, taking into account (3) and (9), we have, for every fixed non-negative integer i,

$$\lim_{n\to\infty} g_{n,i}(x/(2\sqrt{n})) = \frac{\cos(x)}{\sqrt{\pi}} \left(\frac{1}{\varphi(1+2\lambda)} \right)^i. \tag{11}$$

Finally, from (10) to (11) and using Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to\infty} \frac{(-1)^n\sqrt{n}Q_{2n}(x/(2\sqrt{n}))}{2^{2n}n!} = \frac{\cos(x)}{\sqrt{\pi}} \sum_{i=0}^{\infty} \left(\frac{1}{\varphi(1+2\lambda)}\right)^i = \frac{\cos(x)}{\sqrt{\pi}} \frac{\varphi(1+2\lambda)}{\varphi(1+2\lambda)-1}.$$

If m is odd using relation (4), we can proceed as the even case. □

From this theorem we can obtain additional information about zeros of Q_n. We know that these zeros accumulate in $\mathbb{R}$ when $n\to\infty$. Now, we have

Corollary 1. *Let $x_{n,i}$ be the zeros of Q_n. Then*

$$\lim_{n\to\infty} 2\sqrt{n}x_{2n,i} = (2i-1)\frac{\pi}{2}, \quad \lim_{n\to\infty} 2\sqrt{n}x_{2n+1,i} = i\pi, \;\; i\in\mathbb{Z}.$$

Proof. Use Theorem 1 and the Theorem of Hurwitz (see, for example, [7, Theorem 1.91.3, p. 22]). □

Since we have uniform convergence in the result obtained in Theorem 1, we can get asymptotic results for the derivatives of Q_n. In particular, we have

$$\lim_{n\to\infty} \frac{(-1)^n Q'_{2n}(x/(2\sqrt{n}))}{2^{2n+1}n!} = -\Theta(\lambda)\frac{\sin(x)}{\sqrt{\pi}},$$
$$\lim_{n\to\infty} \frac{(-1)^n Q'_{2n+1}(x/(2\sqrt{n}))}{2^{2n+2}\sqrt{n}n!} = \Theta(\lambda)\frac{\cos(x)}{\sqrt{\pi}},$$

both uniformly on compact subsets of $\mathbb{C}$. Thus, we have asymptotic information about the critical points $y_{n,i}$ of Q_n, that is,

$$\lim_{n\to\infty} 2\sqrt{n}y_{2n,i} = i\pi, \quad \lim_{n\to\infty} 2\sqrt{n}y_{2n+1,i} = (2i-1)\frac{\pi}{2}, \;\; i\in\mathbb{Z}.$$

Now, we turn to *Case* II, that is, we consider the Sobolev inner product

$$(f,g)_S = \int f(x)g(x)e^{-x^2}\,dx + \lambda\int f'(x)g'(x)\frac{e^{-x^2}}{x^2+a^2}\,dx, \quad \lambda>0, \;\; a\in\mathbb{R}\backslash\{0\} \tag{12}$$

and let Q_n be the orthogonal polynomials with respect to (12). We get the following result:

Proposition 1. *It holds,*

$$\lim_{n\to\infty} \frac{(-1)^n\sqrt{n}Q_{2n}(x/(2\sqrt{n}))}{2^{2n}n!} = \lim_{n\to\infty} \frac{(-1)^n Q_{2n+1}(x/(2\sqrt{n}))}{2^{2n+1}n!} = 0,$$

uniformly on compact subsets of $\mathbb{C}$.

Proof. We know a relation between these Sobolev orthogonal polynomials and Hermite polynomials (see [2, Lemma 2.5])

$$R_{n+2}(x) := H_{n+2}(x) + \sigma_n \frac{n+2}{n} H_n(x) = Q_{n+2}(x) + \tilde{a}_n Q_n(x), \quad n \geqslant 1, \tag{13}$$

where σ_n are non-zero constants. We also know (see [2, Lemma 2.4])

$$\lim_{n\to\infty} \frac{\sigma_n}{2n} = 1.$$

Thus, using (3)–(4), we can establish that

$$\lim_{n\to\infty} \frac{(-1)^n \sqrt{n} R_{2n}(x/(2\sqrt{n}))}{2^{2n} n!} = \lim_{n\to\infty} \frac{(-1)^n R_{2n+1}(x/(2\sqrt{n}))}{2^{2n+1} n!} = 0, \tag{14}$$

uniformly on compact subsets of $\mathbb{C}$.

Therefore, taking into account Lemma 2.6 in [2], that is, the sequence $\{\tilde{a}_n/(2(n+2))\}$ is uniformly bounded by $(1+a^2)/(1+a^2+2\lambda)$ and

$$\lim_{n\to\infty} \frac{\tilde{a}_n}{2(n+2)} = \frac{1}{\varphi(1+2\lambda)},$$

and using (14), it only remains to proceed as in Theorem 1 in order to obtain the result. □

Remark. The result of Proposition 1 corresponds well with Theorem 2.7 in [2] where it was established that

$$\lim_{n\to\infty} \frac{Q_n(x)}{H_n(x)} = 0,$$

uniformly on compact subsets of $\mathbb{C}\backslash\mathbb{R}$. We think that to improve the result of Proposition 1, it would be necessary to obtain an adequate Mehler–Heine-type formula for the polynomials $R_n(x) = 2^n x^n + \cdots$ which are in some sense very close to the orthogonal polynomials associated with the measure $d\mu_1 = (e^{-x^2}/(x^2+a^2))\,dx$. Obviously, it is not possible to obtain any asymptotic information about the zeros of Q_n from Proposition 1. In Section 4 a conjecture about the zeros of Q_n is done, supported by numerical experiments.

3. Upper bound for $|Q_n|$

We give an upper bound for $|Q_n|$ on $\mathbb{R}$, analogous to that for the Hermite polynomials.

Proposition 2. *It holds*,

(a) *In Case* I,

$$|Q_n(x)| < k e^{x^2/2} 2^{n/2} n!! \, \frac{1 - r^{[n/2]+1}}{1-r}, \quad x \in \mathbb{R},$$

(b) *In Case* II,

$$|Q_n(x)| < k e^{x^2/2} 2^{n/2} n!! \left(1 + \sqrt{5} + \frac{\sqrt{5}}{3} a^2\right) \frac{1 - s^{[n/2]+1}}{1-s}, \quad x \in \mathbb{R}, \ a \neq 0,$$

where $k \simeq 1.086435$, $r = 1/(1+2\lambda)$, $s = (1+a^2)/(1+a^2+2\lambda)$, $n!!$ *denotes the double-factorial of* n, *that is*, $n!! = \prod_{j=0}^{[n/2]-1}(n-2j)$.

Proof. (a) From (7), we have

$$\frac{|Q_n(x)|}{2^{n/2}\sqrt{n!}} \leqslant \sum_{i=0}^{[n/2]} \frac{b_i^{(n)}}{2^i}\sqrt{\frac{(n-2i)!}{n!}}\,\frac{|H_{n-2i}(x)|}{2^{(n/2)-i}\sqrt{(n-2i)!}}$$

$$= \sum_{i=0}^{[n/2]} \frac{b_i^{(n)}}{2^i\sqrt{\prod_{j=0}^{2i-1}(n-j)}}\,\frac{|H_{n-2i}(x)|}{2^{(n/2)-i}\sqrt{(n-2i)!}}.$$

Now, using the relation $|H_n(x)|/(2^{n/2}\sqrt{n!}) < k\mathrm{e}^{x^2/2}$ (see [1, f.(22.14.17) p. 346]) we get

$$\frac{|Q_n(x)|}{2^{n/2}\sqrt{n!}} < k\mathrm{e}^{x^2/2}\sum_{i=0}^{[n/2]} \frac{b_i^{(n)}}{2^i\prod_{j=0}^{i-1}(n-2j)}\sqrt{\frac{\prod_{j=0}^{i-1}(n-2j)^2}{\prod_{j=0}^{2i-1}(n-j)}}$$

$$= k\mathrm{e}^{x^2/2}\sum_{i=0}^{[n/2]} \frac{b_i^{(n)}}{2^i\prod_{j=0}^{i-1}(n-2j)}\sqrt{\prod_{j=0}^{i-1}\frac{n-2j}{n-2j-1}}$$

$$\leqslant k\mathrm{e}^{x^2/2}\sqrt{\prod_{j=0}^{[n/2]-1}\frac{n-2j}{n-2j-1}}\sum_{i=0}^{[n/2]} \frac{b_i^{(n)}}{2^i\prod_{j=0}^{i-1}(n-2j)}.$$

Then, using (see [2, Lemma 2.2])

$$\frac{b_i^{(n)}}{2^i\prod_{j=0}^{i-1}(n-2j)} = \prod_{j=1}^{i}\frac{a_{n-2j}}{2(n-2j+2)} < \left(\frac{1}{1+2\lambda}\right)^i = r^i$$

we get

$$\frac{|Q_n(x)|}{2^{n/2}\sqrt{n!}} < k\mathrm{e}^{x^2/2}\sqrt{\frac{n!!}{(n-1)!!}}\,\frac{1-r^{[n/2]+1}}{1-r}.$$

It only remains to use $\sqrt{n!!/(n-1)!!}\sqrt{n!} = n!!$

(b) Indeed, relation (13) can be rewritten as

$$R_n(x) = Q_n + \tilde{a}_{n-2}Q_{n-2}, \quad n \geqslant 0, \tag{15}$$

where $\tilde{a}_{-2} = \tilde{a}_{-1} = \tilde{a}_0 = 0$ being $R_i(x) = H_i(x)$, $i = 0, 1, 2$, and $\tilde{a}_n = \sigma_n((n+2)/n)(k_n/\tilde{k}_n)$, $n \geqslant 1$ (see [2, Lemma 2.5]). Thus, applying (15) in a recursive way we obtain

$$Q_m(x) = \sum_{i=0}^{[m/2]} (-1)^i \tilde{b}_i^{(m)} R_{m-2i}, \quad m \geqslant 0,$$

where $\tilde{b}_i^{(m)} = \prod_{j=1}^{i} \tilde{a}_{m-2j}$, $i \geqslant 1$ and $\tilde{b}_0^{(m)} = 1$.

Therefore, as in (a), we have

$$\frac{|Q_n(x)|}{2^{n/2}\sqrt{n!}} \leqslant \sum_{i=0}^{[n/2]} \frac{\tilde{b}_i^{(n)}}{2^i \sqrt{\prod_{j=0}^{2i-1}(n-j)}} \frac{|R_{n-2i}(x)|}{2^{(n/2)-i}\sqrt{(n-2i)!}}.$$

On the other hand,

$$R_n(x) = H_n(x) + \sigma_{n-2}\frac{n}{n-2}H_{n-2}, \quad n \geqslant 3,$$

where (see [2, f.(2.12)])

$$\sigma_n = \frac{k'_{n+1}}{4k_{n-1}} > 0, \quad n \geqslant 1, \text{ with } k'_n = \int_{-\infty}^{\infty} T_n^2(x) \frac{\mathrm{e}^{-x^2}}{x^2 + a^2}\,\mathrm{d}x,$$

being T_n the orthogonal polynomials with respect to the inner product $(f, g) = \int_{-\infty}^{\infty} f(x)g(x)(\mathrm{e}^{-x^2}/(x^2 + a^2))\,\mathrm{d}x$ and with the same leading coefficient as H_n.

We know (see [2, f.(2.19)]) that

$$\sigma_{n+1} + \frac{4n(n-1)}{\sigma_{n-1}} = 4(n + a^2) + 2, \quad n \geqslant 2, \tag{16}$$

then $\sigma_{n+1} < 4(n + a^2) + 2$ for $n \geqslant 2$. Thus, for $n \geqslant 5$,

$$\frac{|R_n(x)|}{2^{n/2}\sqrt{n!}} \leqslant \frac{|H_n(x)|}{2^{n/2}\sqrt{n!}} + \frac{\sigma_{n-2}}{2(n-2)}\sqrt{\frac{n}{n-1}}\frac{|H_{n-2}(x)|}{2^{n/2-1}\sqrt{(n-2)!}}$$

$$< k\mathrm{e}^{x^2/2}\left(1 + \frac{\sigma_{n-2}}{2(n-2)}\sqrt{\frac{n}{n-1}}\right) < k\mathrm{e}^{x^2/2}\left(1 + \sqrt{5} + \frac{\sqrt{5}}{3}a^2\right).$$

Since $R_4(x) = H_4(x) + 2\sigma_2 H_2(x)$ and $R_3(x) = H_3(x) + 3\sigma_1 H_1(x)$, and straightforward computations show that $\sigma_1 < 2$ and $\sigma_2 < 4$, we have that, for $n \geqslant 0$, it holds

$$\frac{|R_n(x)|}{2^{n/2}\sqrt{n!}} < k\mathrm{e}^{x^2/2}\left(1 + \sqrt{5} + \frac{\sqrt{5}}{3}a^2\right).$$

Now, in order to obtain the result, we can proceed exactly as in (a) taking into account that (see [2, Lemma 2.6]):

$$\frac{\tilde{b}_i^{(n)}}{2^i \prod_{j=0}^{i-1}(n-2j)} = \prod_{j=1}^{i} \frac{\tilde{a}_{n-2j}}{2(n-2j+2)} < \left(\frac{1+a^2}{1+a^2+2\lambda}\right)^i := s^i. \qquad \square$$

4. Numerical examples and remarks

We illustrate Corollary 1 with two numerical examples. We compare the limit values $(2i-1)\pi/2$ and $i\pi$, where $i=1,2,3,4$, with the first four positive real zeros of Q_{2n} and Q_{2n+1} rescaled by the factor $2\sqrt{n}$, respectively, for $n=25,50,75,100$. Note that Q_n are symmetric, that is, $Q_n(-x)=(-1)^n Q_n(x)$. In order to obtain the numerical results, we use relation (6) and the recurrence relation for the coefficients a_n in (6) given by (see [2, f.(2.3)]):

$$a_n = \frac{4(n+1)(n+2)}{2(2(2\lambda+1)n+1+2a^2)-a_{n-2}}, \quad n \geqslant 2$$

with

$$a_0 = \frac{4}{1+2a^2}, \quad a_1 = \frac{12}{3+2a^2+4\lambda}.$$

First example: $a=0$ and $\lambda=1$.

$2\sqrt{n}x_{2n,i}$	$i=1$	$i=2$	$i=3$	$i=4$
$n=25$	1.5698692268	4.7111057199	7.8568557403	11.0101937059
$n=50$	1.5702245533	4.7110548442	7.8530299109	10.9969170269
$n=75$	1.5703920503	4.7113465231	7.8528123556	10.9951312645
$n=100$	1.5704845829	4.7115498501	7.8529034979	10.9947380373
Limit value $(2i-1)\pi/2$	**1.5707963268**	**4.7123889804**	**7.8539816340**	**10.9955742876**

$2\sqrt{n}x_{2n+1,i}$	$i=1$	$i=2$	$i=3$	$i=4$
$n=25$	3.1091797287	6.2212173042	9.3390062419	12.4655126418
$n=50$	3.1249658267	6.2506757089	9.3778760451	12.5073179403
$n=75$	3.1304135054	6.2611622256	9.3925818469	12.5250090005
$n=100$	3.1331726373	6.2665351221	9.4002774525	12.5345899285
Limit value $i\pi$	**3.1415926536**	**6.2831853072**	**9.4247779608**	**12.5663706144**

Second example: $a=16.25$ and $\lambda=7.2$.

$2\sqrt{n}x_{2n,i}$	$i=1$	$i=2$	$i=3$	$i=4$
$n=25$	1.5638248280	4.6929891583	7.8267166702	10.9681140574
$n=50$	1.5673453033	4.7024188792	7.8386425814	10.9767872318
$n=75$	1.5685056399	4.7056877955	7.8433828210	10.9819334377
$n=100$	1.5690824506	4.7073436604	7.8458938729	10.9849260131
Limit value $(2i-1)\frac{\pi}{2}$	**1.5707963268**	**4.7123889804**	**7.8539816340**	**10.9955742876**

$2\sqrt{n}x_{2n+1,i}$	$i=1$	$i=2$	$i=3$	$i=4$
$n=25$	3.0974699203	6.1978284839	9.3039995611	12.4189787398
$n=50$	3.1192959362	6.2393393699	9.3608801311	12.4846727358
$n=75$	3.1266791037	6.2536944055	9.3813825738	12.5100812193
$n=100$	3.1303897036	6.2609696616	9.3919302800	12.5234622650
Limit value $i\pi$	**3.1415926536**	**6.2831853072**	**9.4247779608**	**12.5663706144**

We also give a numerical example about critical points of Q_n. For example, we take $a=16.25$ and $\lambda=7.2$.

$2\sqrt{n}y_{2n,i}$	$i=1$	$i=2$	$i=3$	$i=4$
$n=25$	3.1594830497	6.3221561784	9.4912516037	12.6700874601
$n=50$	3.1505015656	6.3017887843	9.4546498604	12.6098781249
$n=75$	3.1475288458	6.2954052502	9.4439772711	12.5935939674
$n=100$	3.1460444775	6.2922840149	9.4389138295	12.5861294535
Limit value $i\pi$	**3.1415926536**	**6.2831853072**	**9.4247779608**	**12.5663706144**

$2\sqrt{n}y_{2n+1,i}$	$i=1$	$i=2$	$i=3$	$i=4$
$n=25$	1.5638124248	4.6929519452	7.8266546359	10.9680271832
$n=50$	1.5673412691	4.7024067765	7.8386224094	10.9767589897
$n=75$	1.5685036668	4.7056818763	7.8433729554	10.9819196256
$n=100$	1.5690812843	4.7073401616	7.8458880415	10.9849178491
Limit value $(2i-1)\pi/2$	**1.5707963268**	**4.7123889804**	**7.8539816340**	**10.9955742876**

Finally, we turn to *Case* II again. In order to obtain some light about the asymptotic behavior of the small zeros of Q_n in this case (see Remark after Proposition 1), we have done some numerical experiments. For the computations, we have used relations (13) and (16), and the recurrence relation for $\tilde{a}_n$ (see [2, f.(2.15)]):

$$\tilde{a}_n=\frac{((n+2)/n)k_n\sigma_n}{k_n+n^2k_{n-2}(\sigma_{n-2}/(n-2))^2+16\lambda n^2k_{n-3}\sigma_{n-2}-nk_{n-2}(\sigma_{n-2}/(n-2))\tilde{a}_{n-2}},\quad n\geqslant 3.$$

We denote by $x_{n,i}$ and $t_{n,i}$ the positive zeros of Q_n and T_n, respectively. Note that, as in the proof of (b) in Proposition 2, T_n are the orthogonal polynomials associated to the measure $\mathrm{d}\mu_1=(\mathrm{e}^{-x^2}/(x^2+a^2))\,\mathrm{d}x$, $a\in\mathbb{R}\backslash\{0\}$, with the same leading coefficient as $H_n(x)$, $Q_n(-x)=(-1)^nQ_n(x)$ and $T_n(-x)=(-1)^nT_n(x)$. We have done several numerical experiments. Here, we show one of them where we have chosen $a=1.5$ and $\lambda=2$, obtaining then, the following results:

	$2\sqrt{[n/2]}x_{n,1}$	$2\sqrt{[n/2]}t_{n,1}$	$2\sqrt{[n/2]}x_{n,2}$	$2\sqrt{[n/2]}t_{n,2}$
$n=10$	1.4027323418	1.3987941473	4.2947841151	4.2696568868
$n=15$	2.7666414312	2.7634271573	5.6196438529	5.6010602493
$n=20$	1.4410241107	1.4413451552	4.3549625473	4.3510453485
$n=25$	2.8517688149	2.8521565689	5.7457770580	5.7407669190

	$2\sqrt{[n/2]}x_{n,3}$	$2\sqrt{[n/2]}t_{n,3}$	$2\sqrt{[n/2]}x_{n,4}$	$2\sqrt{[n/2]}t_{n,4}$
$n=10$	7.4013947771	7.3354957604	10.8326652093	10.7188917547
$n=15$	8.6078598933	8.5626160809	11.7639582963	11.6870682334
$n=20$	7.3487596798	7.3309838280	10.4453062669	10.4063028345
$n=25$	8.7076452460	8.6902950927	11.7481557198	11.7136768380

Conjecture. After these experiments, we conjecture that the zeros of Q_n are in some sense close to those of T_n.

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References

[1] M. Abramowitz, I.A. Stegun, Pocketbook of Mathematical Functions, Verlag, Harri Deutsch, 1984.
[2] M. Alfaro, J.J. Moreno–Balcázar, T.E. Pérez, M.A. Piñar, M.L. Rezola, Asymptotics of Sobolev orthogonal polynomial for Hermite coherent pairs, J. Comp. Appl. Math. 113 (2001) 141–150.
[3] A. Iserles, P.E. Koch, S.P. Nørsett, J.M. Sanz–Serna, On polynomials orthogonal with respect to certain Sobolev inner products, J. Approx. Theory 65 (2) (1991) 151–175.
[4] D.C. Lewis, Polynomial least square approximations, Amer. J. Math. 69 (1947) 273–278.
[5] F. Marcellán, J.J. Moreno–Balcázar, Strong and Plancherel–Rotach asymptotics of non-diagonal Laguerre–Sobolev orthogonal polynomials, J. Approx. Theory 110 (2001) 54–73.
[6] H.G. Meijer, Determination of all coherent pairs, J. Approx. Theory 89 (3) (1997) 321–343.
[7] G. Szegő, Orthogonal Polynomials, 4th Edition, Amer. Math. Soc. Colloq. Publ., Vol. 23, American Mathematical Society, Providence, RI, 1975.